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Heat transfer from a hot film in reversing shear flow

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The two-dimensional thermal boundary layer over a finite hot film embedded in a plane insulating wall, with a shear flow over it which reverses its direction, is analysed approximately using methods similar to those previously developed for viscous boundary layers (Pedley 1976). The heat transfer from the film is calculated both for uniformly decelerated and for oscillatory wall shear, and application is made to predict the response of hot-film anemometers actually used to measure oscillatory velocities in water and blood. The results predict that the velocity amplitude measured on the assumption of a quasi-steady response will depart from the actual amplitude at values of the frequency parameter St greater than about 0·3 ($St = \Omega X_0/U_0$, where Ω = frequency, U_0 = mean velocity, X_0 = distance of hot film from the leading edge of the probe). This is in good agreement with experiment. So too is the shape of the predicted anemometer output as a function of time throughout a complete cycle, for cases when the response is not quasi-steady. However, there is a significant phase lead between the predicted and the experimental outputs. Various possible reasons for this are discussed; no firm conclusions are reached, but the most probable cause lies in the three-dimensionality of the velocity and temperature fields, since the experimental hot films are only about 2·5 times as broad as they are long, and are mounted on a cylinder not a flat plate.

1. Introduction

When a metallic film flush with the surface of an insulated solid boundary is heated to a temperature somewhat greater than that of the surrounding fluid medium, and when that fluid flows past the boundary, the rate of heat loss from the film is related to the flow. If the flow is steady, if the forced heat transfer greatly exceeds any free convective heat transfer, if the geometry of the film is such that two-dimensional boundary-layer theory can be applied to the thermal boundary layer on the film, and if the thickness of this thermal boundary layer is sufficiently small compared with the lateral distance over which the fluid velocity gradient, or shear, varies, then both theory and experiment show that the rate of heat transfer is directly proportional to the $\frac{1}{3}$-power of the value of the shear at the wall (Liepmann & Skinner 1954). Thus the hot film can be used to measure wall shear. Furthermore, if the film is embedded in the surface of a small probe which can be inserted into a stream of flowing fluid, then (again assuming that steady two-dimensional boundary-layer theory is applicable) the wall shear on the film is proportional to the $\frac{3}{2}$-power of the stream velocity. Hence the heat transfer is proportional to the $\frac{1}{2}$-power of the stream velocity, and the device can be used to measure velocity.

In unsteady flow, however, the variations in the wall shear have a larger relative amplitude than those in the stream velocity, and also have a phase lead over it. Also the heat transfer lags the wall shear. Furthermore if the flow velocity comes close to zero or reverses its direction, the viscous boundary layer cannot respond in a quasi-steady manner, and if the wall shear comes close to zero or reverses its direction (which can happen with a unidirectional free stream) the thermal boundary layer cannot be quasi-steady. In particular, the heat transfer is positive whatever the direction of wall shear, and has a minimum value well above zero when the shear passes through zero. Hence to use the steady calibration curve in measuring unsteady velocities is potentially dangerous, and careful unsteady calibration experiments are needed. For the hot-film anemometers which have been used to measure blood velocities in large arteries, such calibrations have been performed in sinusoidally oscillating flow by Seed & Wood (1970) and by Clark (1974). Both studies showed that the probes responded quasi-steadily until the frequency parameter $\Omega X_0/U_0$ exceeded a number of the order of 0·3 (here Ω is the angular frequency of the oscillations, X_0 is the distance of the hot film from the leading edge of the probe and U_0 is the mean velocity of the free stream).

A theoretical analysis of the response of such probes was performed by Pedley (1972*a*) for the case where neither the free-stream velocity nor the wall shear came close to zero. However, the early reversal of wall shear renders the theory inapplicable to the more interesting experiments in which large departures from quasi-steady behaviour are observed, and the results therefore have little practical value. It is the purpose of the present paper to extend the theory of the thermal boundary layer to cases in which the wall shear reverses its direction or comes close to zero. The methods will be similar to those already developed for viscous boundary layers in reversing flow (Pedley 1976), but certain details are different and have to be examined carefully.

The (approximate) theory is developed in the next section, and in § 3 it is applied to the case of a uniformly decelerating shear which reverses its direction at time $\hat{t} = 0$. The results for this case are used to guide the choice of previously undetermined constants, which are kept fixed subsequently. In § 3 the theory is also applied to predict the heat transfer in a sinusoidally oscillating shear with non-zero mean, and the results are expected to be applicable to the use of hot films for measuring wall shear in oscillating pipe flow. In § 4 the theory is applied to the hot-film anemometers used in sinusoidally oscillating free streams by Seed & Wood (1970) and by Clark (1974); in this case it is necessary first to calculate the wall shear, by a direct application of the theory of Pedley (1976). The *shape* of the predicted curve of heat transfer against time agrees very well with experiment, but its *phase* shows a lead of approximately $\frac{1}{4}\pi$ over the experimental curve (in the experiments the heat transfer lags behind the free stream, while in the theory it leads the free stream). Such a phase lead was also observed by Pedley (1972*a*), and is therefore not an aberration introduced by the approximations required to deal with reversing shear, but becomes apparent in the first departure from quasi-steady behaviour. The assumptions involved in applying the present theory to the hot-film velocity probes are examined closely in § 5, and it is concluded that the most probable cause of the unwanted phase lead lies in the three-dimensional nature of the velocity and temperature fields over the hot film, because the probe is not mounted on a flat plate but on a cylinder, and because diffusion in the cross-stream direction tends to increase the 'thermal inertia' of the probe.

2. Development of the approximate theory

The heated film is modelled as the region $0 \leqslant \hat{x} \leqslant l$ of the solid plane $\hat{y} = 0$, of which the remainder allows no heat to pass, and the fluid velocity over it is taken to be in the $\hat{x}$ direction and equal to $\hat{S}(\hat{t})\hat{y}$, where $\hat{S}$ is the wall shear rate (dependent on time $\hat{t}$) and a caret denotes a dimensional quantity. The temperature T_1 of the film is taken to exceed the temperature T_0 of the oncoming fluid. The object of the theory is to calculate the temperature T everywhere over the film, and hence to deduce the rate of heat loss from the film. $\hat{S}$ is taken to reverse direction at $\hat{t} = 0$, being positive for $\hat{t} < 0$. We suppose that, long before $\hat{t} = 0$, the Péclet number $l^2\hat{S}/\kappa$ (where κ is the uniform thermal diffusivity of the fluid) is large and $\hat{S}$ not very rapidly varying, so that there is a thin, approximately quasi-steady thermal boundary layer over the film, growing from the leading edge $\hat{x} = 0$. Similarly, long after $\hat{t} = 0$, we expect there to be a quasi-steady thermal boundary layer growing from the trailing edge $\hat{x} = l$. The present theory is intended to give an approximate description of the transition between these two states.

We introduce dimensionless variables as follows:

$$t = \hat{t}/\hat{t}_0, \quad x = \hat{x}/l, \quad y = \hat{y}/(\hat{t}_0\,\kappa)^{\frac{1}{2}}, \quad S(t) = \hat{S}(\hat{t})/S_0, \\ \theta = (T-T_0)/(T_1-T_0), \tag{1}$$

where $\hat{t}_0$ and S_0 are appropriate time and shear scales, to be chosen according to the application. The temperature equation then reduces to

$$\theta_t + \beta S(t)\, y\theta_x = \theta_{yy}, \tag{2}$$

where

$$\beta = S_0 \hat{t}_0^{\frac{3}{2}} \kappa^{\frac{1}{2}}/l, \tag{3}$$

a dimensionless parameter. In the application to the uniformly decelerated and the sinusoidally oscillating wall shear (§ 3), we choose $\hat{t}_0 = (l^2/S_0^2\,\kappa)^{\frac{1}{3}}$, so that $\beta = 1$ and the scale for the thermal-boundary-layer thickness, $(\hat{t}_0\,\kappa)^{\frac{1}{2}}$, is equal to $(l\kappa/S_0)^{\frac{1}{3}}$ as in the steady case (Lévêque 1928). In the application to the hot-film anemometer (§ 4) a different choice for $\hat{t}_0$ is more convenient. The boundary conditions on θ are

$$\theta = 1 \quad \text{on} \quad y = 0, \quad \theta \to 0 \quad \text{as} \quad y \to \infty.$$

The approximate solution is constructed in the same way as for the viscous case (Pedley 1976). Thus we assert that at every point on the film (every x)

there is a time $t = -t_1(x)$ before which the temperature distribution is approximately quasi-steady, representing a balance between convection and diffusion with only a small correction for the θ_t tèrm in (2). We take the solution for $t < -t_1$ to be given by the first two terms of a series whose leading term is exactly quasi-steady; this series was developed by Pedley (1972*a*). We further assert that there is a time $t_2(x)$ such that for $t > t_2$ the solution is given by the same two terms, except that the leading edge is at $x = 1$ not $x = 0$. For intermediate times, during which $S(t)$ passes through zero, diffusion will proceed, but convection will be relatively unimportant; nevertheless we expect the boundary-layer approximation still to be applicable. We assert that the temperature is given by a solution of (2) without the middle convective term. The big approximation of the theory is the assertion that the transitions between almost quasi-steady and purely diffusive solutions take place abruptly, not gradually.

The full approximate solution for θ (in the case $\beta = 1$) is thus given by (4)–(6), as follows:

$$\theta = \theta_0(\eta_1) + 3^{\frac{1}{3}} x^{\frac{2}{3}} S^{-\frac{5}{3}}(t)\,\dot{S}(t)\,\theta_1(\eta_1) \quad \text{for} \quad t \leqslant -t_1(x), \tag{4}$$

where
$$\eta_1 = y[S(t)/9x]^{\frac{1}{3}},$$

$$\theta_0(\eta_1) \equiv 1 - C\int_0^{\eta_1} e^{-\eta^3}\,d\eta, \quad C = 1/\Gamma(\tfrac{4}{3}) = 1{\cdot}120,$$

and θ_1 satisfies

$$\theta_1'' + 3\eta_1^2\,\theta_1' - 6\eta_1\,\theta_1 = -C\eta_1 \exp(-\eta_1^3), \quad \theta_1(0) = \theta_1(\infty) = 0;$$

$$\theta = \theta_0(\eta_2) - 3^{\frac{1}{3}}(1-x)^{\frac{2}{3}}[-S(t)]^{\frac{5}{3}}\,\dot{S}(t)\,\theta_1(\eta_2) \quad \text{for} \quad t \geqslant +t_2(x), \tag{5}$$

where
$$\eta_2 = y[-S(t)/9(1-x)]^{\frac{1}{3}};$$

$$\theta = \operatorname{erfc}\eta_0 \quad \text{for} \quad -t_1(x) < t < t_2(x), \tag{6}$$

where
$$\eta_0 = \tfrac{1}{2}y[t + t_0(x)]^{-\frac{1}{2}}$$

and $-t_0(x)$ is a virtual origin of the diffusive solution which must be determined along with $t_1(x)$ and $t_2(x)$.

The choice of take-over times $-t_1$ and t_2 is more difficult than in the viscous case and the simple thermal case of Pedley (1975). There it was argued that the diffusive solution would take over from the initial quasi-steady solution at a given value of x when the influence of the leading edge ceased to be felt there, i.e. when fluid particles which had passed the leading edge first failed to arrive at x before being swept away by the reversing flow. Similarly the new quasi-steady solution would take over from the diffusive solution when fluid particles which had passed the new leading edge first arrived at x. Thus, for example, t_2 was given by

$$1 - x = \int_0^{t_2} U(t)\,dt,$$

where $U(t)$ was the free-stream velocity. In the present case, however, there is no unique free-stream velocity because the flow consists of a uniform shear and the velocity is proportional to y. In order to apply a similar condition we must fix the convection velocity by picking a particular value of y. It would seem sensible to choose a value in some way representative of the boundary-layer thickness at the take-over time. We therefore pick y such that η_1(or η_2) takes a particular value $\bar{\eta}_1$ (or $\bar{\eta}_2$) at position x and time $-t_1$ (or $+t_2$); the values to be used for $\bar{\eta}_1$ and $\bar{\eta}_2$ are discussed later. From these considerations, then, we determine t_1 to be the solution of

$$x = y\int_{-t_1}^{0} S(t)\,dt \quad \text{when} \quad y = \bar{\eta}_1[9x/S(-t_1)]^{\frac{1}{3}},$$

i.e. of

$$[\tfrac{1}{9}x^2S(-t_1)]^{\frac{1}{3}} = \bar{\eta}_1\int_{-t_1}^{0} S(t)\,dt. \tag{7}$$

Similarly

$$[-\tfrac{1}{9}(1-x)^2S(t_2)]^{\frac{1}{3}} = \bar{\eta}_2\int_{0}^{t_2}[-S(t)]\,dt. \tag{8}$$

The two-term series solutions for $t < -t_1$ and $t > t_2$ (equations (4) and (5)) were shown by Pedley (1972*a*) to be accurate (i.e. differing by only a little from the three-term expansion) only if the coefficient of $\theta_1(\eta_1)$ [or $\theta_1(\eta_2)$], which we denote by λ_1 (or λ_2), remains sufficiently small. A precise condition was not derived, but examination of the results of that paper (translated into the notation of this) suggests that accuracy will be adequate as long as $|\lambda_1|$ (or $|\lambda_2|$) is less than about 0·5,† where

$$\left.\begin{aligned} \lambda_1(t) &= 3^{\frac{1}{3}}x^{\frac{2}{3}}S^{-\frac{5}{3}}(t)\,\dot{S}(t),\\ \lambda_2(t) &= 3^{\frac{1}{3}}(1-x)^{\frac{2}{3}}[-S(t)]^{-\frac{5}{3}}[-\dot{S}(t)].\end{aligned}\right\} \tag{9}$$

(The parameters $|\lambda_1|$ and $|\lambda_2|$ are comparable to the quantities ϵ_1 and ϵ_2 defined by Pedley (1976) for the viscous boundary layer.) This criterion can be used to check the values of t_1 and t_2 given by (7) and (8): if the value of $|\lambda_1(-t_1)|$ or $|\lambda_2(t_2)|$ greatly exceeds 0·5, then the series solutions will be inaccurate for values of t close to $-t_1$ or t_2; however, if they are much less than 0·5, then the diffusive solution will probably be less accurate than the (rejected) series solution. Indeed, for cases in which $S(t)$ comes close to zero without reversing, so that no value of t_1 can be obtained from (7), we use the criterion $|\lambda_1(-t_1)| = 0{\cdot}5$ to determine t_1 ($t = 0$ is then the time of minimum shear). Little would probably be lost by using (9) to determine t_1 and t_2 in general, but the proposed method has a sounder physical basis.

We still have to determine the values of $\bar{\eta}_1$ and $\bar{\eta}_2$ used in the definitions of t_1 and t_2 (equations (7) and (8)), as well as the virtual origin $-t_0(x)$ of the diffusive part of the solution. The latter must be chosen to ensure some continuity at $t = -t_1$ between the diffusive solution and the approximately quasi-steady one from which it takes over. In the viscous case the most fundamental single parameter representing the boundary layer is the displacement thickness, continuity of which ensures continuity of mass flux deficit. In the present case the choice of t_0 is bound up with the choice of $\bar{\eta}_1$. A self-consistent way of choosing both is to require that the 'centre of mass' of the temperature distribution be

† The critical value of 0·5 is derived from the observation on p. 338 of Pedley (1972*a*), that, for the case $S = 1+\alpha\sin\omega t$, inaccuracy first begins to appear close to $\omega t = \frac{3}{2}\pi$ when $\alpha = 0{\cdot}5$ and when ϵ, equal to $3^{\frac{4}{3}}\omega x^{\frac{2}{3}}$ in the present notation, is given the value 2·0. The maximum value of $|\lambda_1|$ in that case is actually 0·49.

continuous at $t = -t_1$, and then the natural choice of $\bar{\eta}_1$ is the value of η_1 at the centre of mass. This is the choice we adopt. The centre of mass $\bar{y}$ is defined by

$$\bar{y} = \int_0^\infty y\theta\, dy \bigg/ \int_0^\infty \theta\, dy$$
$$= \delta \int_0^\infty \eta\theta\, d\eta \bigg/ \int_0^\infty \theta\, d\eta \quad \text{when} \quad \eta = y/\delta.$$

From (4) and (6), continuity of this quantity requires that

$$\tfrac{1}{4}\pi(t_0 - t_1) = [9x/S(-(t_1)]^{\frac{2}{3}}\, \bar{\eta}_1^2, \tag{10a}$$

where

$$\bar{\eta}_1 = \frac{C_0 + \lambda_1(-t_1)\, C_2}{C_1 + \lambda_1(-t_1)\, C_3} \tag{11}$$

and

$$C_0 = \int_0^\infty \eta_1\, \theta_0(\eta_1)\, d\eta_1 = 0{\cdot}187,$$
$$C_1 = \int_0^\infty \theta_0(\eta_1)\, d\eta_1 = 0{\cdot}505,$$
$$C_2 = \int_0^\infty \eta_1\, \theta_1(\eta_1)\, d\eta_1 = 0{\cdot}00302,$$
$$C_3 = \int_0^\infty \theta_1(\eta_1)\, d\eta_1 = 0{\cdot}0451,$$

all correct to three significant figures. Note that $\bar{\eta}_1$ itself depends on x and t_1 through λ_1 (equation (9)).

An alternative choice of t_0, which however gives less guidance on a suitable choice for $\bar{\eta}_1$, is obtained by making the longitudinal heat flux continuous. This is physically reasonable because, if it is not continuous, there is a singularity in the temperature gradient on the hot film which could (if care were not taken to exclude it) result in significant errors in the predicted overall heat-transfer rate from the film. This would require that

$$\int_0^\infty y\theta\, dy$$

be continuous, and hence, from (4) and (6), that

$$t_0 - t_1 = [9x/S(-t_1)]^{\frac{2}{3}}\, [C_0 + \lambda_1(-t_1)\, C_2]. \tag{10b}$$

A third possible choice would be to make

$$\int_0^\infty \theta\, dy$$

continuous, in which case

$$4\pi^{-1}(t_0 - t_1) = [9x/S(-t_1)]^{\frac{2}{3}}\, [C_1 + \lambda_1(-t_1)\, C_3]^2. \tag{10c}$$

The differences in the predictions of heat transfer resulting from these three choices of t_0 are examined for the simple case of a uniformly decelerating shear in the next section, and are shown to be small. Choices other than (11) for $\bar{\eta}_1$

(and the equivalent expression for $\bar{\eta}_2$) are also examined, and these make a considerable difference. However, in the simplest example, (11) gives a value of $\lambda_1(-t_1)$ close to 0·5 (while the other choices do not), and (11) is the only choice with *a priori* justification, so it is retained.

The object of the theory is to calculate the overall time-dependent heat transfer from the film, proportional (in dimensionless terms) to

$$Q(t) = \int_0^1 [-\theta_y]_{y=0}\, dx. \tag{12}$$

The integrand is given by the solutions (4)–(6) to be

$$-\theta_y|_{y=0} = \begin{cases} [S(t)/9x]^{\frac{1}{3}}\,[C-\lambda_1\theta_1'(0)] \quad \text{for} \quad t \leqslant -t_1, & (13a) \\ [-S(t)/9(1-x)]^{\frac{1}{3}}\,[C-\lambda_2\theta_1'(0)] \quad \text{for} \quad t \geqslant t_2, & (13b) \\ [\pi(t_0+t_1)]^{-\frac{1}{2}} \quad \text{for} \quad -t_1 < t < t_2, & (13c) \end{cases}$$

where $\theta_1'(0) = 0{\cdot}143$. At each value of t, the value of x, say x_T, at which there is transition between a quasi-steady and the diffusive solution must be calculated if it exists, and Q evaluated by integrating *either* (13*a*) from 0 to x_T and (13*c*) from x_T to 1, if $S(t) > 0$, *or* (13*c*) from 0 to x_T and (13*b*) from x_T to 1, if $S(t) < 0$.

3. Heat transfer with given wall shear variation

Uniform deceleration

In order to test various possibilities for $\bar{\eta}_1$, $\bar{\eta}_2$ and t_0, we begin by considering the simplest possible form of $S(t)$, a uniform deceleration through zero, i.e.

$$S(t) = -t.$$

[In dimensional terms this corresponds to a velocity $\hat{u} = -A\hat{t}\hat{y}$, the shear and time scales S_0 and $\hat{t}_0$ of (1) being given by $S_0 = (l^2A^3/\kappa)^{\frac{1}{3}}$ and $\hat{t}_0 = (l^2/A^2\kappa)^{\frac{1}{3}}$.] In this case (7) and (8) give

$$t_1 = (\tfrac{1}{3}x)^{\frac{2}{5}}\,(2/\bar{\eta}_1)^{\frac{6}{5}}, \quad t_2 = [\tfrac{1}{3}(1-x)]^{\frac{2}{5}}\,(2/\bar{\eta}_2)^{\frac{6}{5}}, \tag{14}$$

whence (9) gives

$$\lambda_1(-t_1) = -1{\cdot}5\bar{\eta}_1, \quad \lambda_2(t_2) = +1{\cdot}5\bar{\eta}_2. \tag{15}$$

If $\bar{\eta}_1$ is given by (11), and $\bar{\eta}_2$ by a similar equation involving λ_2, substitution of (15) yields $-\lambda_1 = 0{\cdot}58$ and $\lambda_2 = 0{\cdot}53$, which are close to the value of 0·5 above which the approximately quasi-steady solutions become inaccurate. If the second term in the quasi-steady solutions (4) and (5) is ignored (and hence the second terms in the numerator and denominator of (11) are also ignored) then $-\lambda_1(-t_1)$ and $\lambda_2(t_2)$ are both equal to 0·55, suggesting that the second term does not make a large difference; the heat-transfer calculations will confirm this. If $\bar{\eta}_1$ and $\bar{\eta}_2$ are chosen to be somewhere other than at the centre of mass of the temperature distribution, then $-\lambda_1(-t_1)$ and $\lambda_2(t_2)$ will take different values: if $\bar{\eta}_1$ and $\bar{\eta}_2$ are taken to be further out, $-t_1$ and t_2 are smaller, and the values of $-\lambda_1$ and λ_2 at take-over increase, thereby presumably decreasing the accuracy. If $\bar{\eta}_1$ and $\bar{\eta}_2$ are taken closer to the wall, the values of $-\lambda_1$ and λ_2 decrease.

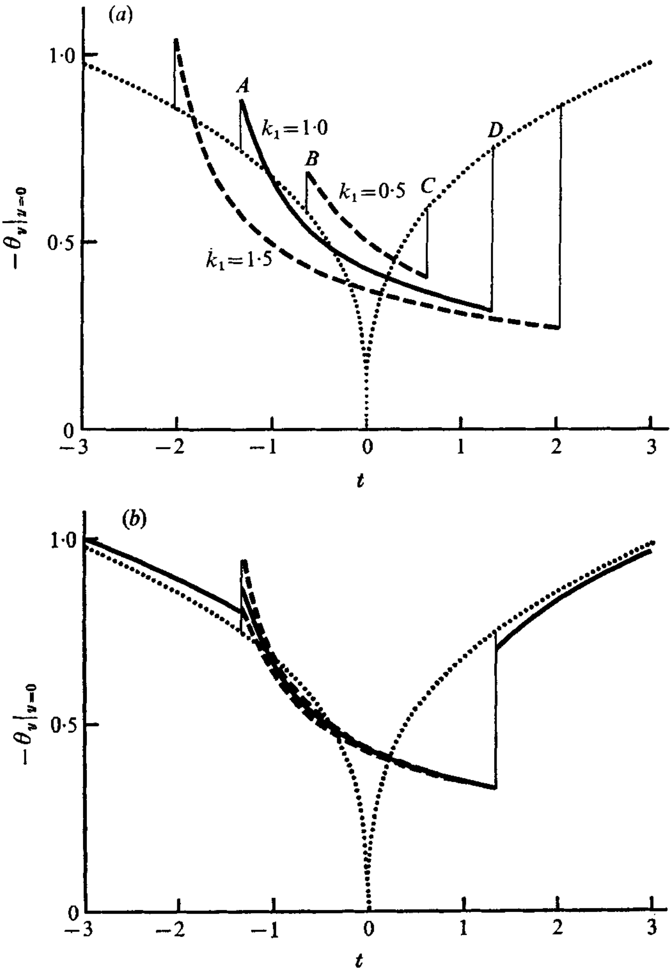


FIGURE 1. Dimensionless heat transfer per unit length $(-\theta_y|_{y=0})$ as a function of time t for uniformly decelerating shear. Dotted curve is quasi-steady in each case. (*a*) Diffusive solutions compared for three different values of k_1(equation (16)); solid curve ($k_1 = 1{\cdot}0$) is the 'standard' case. (*b*) Diffusive solutions compared for three different values of k_2 (equation (17)). Solid curve represents 'standard' case ($k_2 = 0{\cdot}175$), including two terms of the approximately quasi-steady expansions. Broken curves: upper, $k_2 = 0{\cdot}15$; lower, $k_2 = 0{\cdot}2$.

Three cases have been examined numerically to test the effect of such changes on the predicted heat transfer; in each case the second terms in the series solutions (4) and (5) were ignored for convenience. Let the value of t_1 given by (14) with $\overline{\eta}_1$ equal to C_0/C_1 (see (11)) be $\bar{t}_1$; then the three cases taken were

$$t_1 = k_1 \bar{t}_1 \quad \text{with} \quad k_1 = 0{\cdot}5,\ 1{\cdot}0,\ 1{\cdot}5 \tag{16}$$

(when $k_1 = 0{\cdot}5$, $\overline{\eta}_1 \approx 3{\cdot}2C_0/C_1$ and $-\lambda_1$ is increased by a similar amount; when $k_1 = 1{\cdot}5$, $\overline{\eta}_1 \approx 0{\cdot}5C_0/C_1$). The second take-over time t_2 was also changed by the same factor.

The numerical solutions for this shear variation also have to distinguish

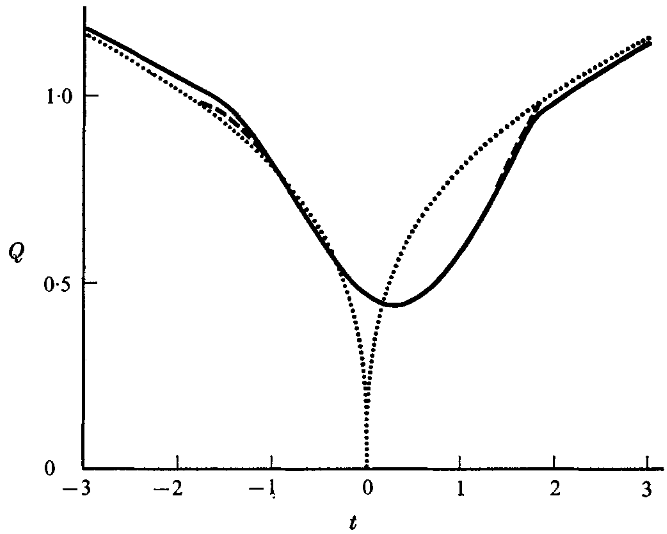


FIGURE 2. Dimensionless heat transfer Q from the whole film as a function of t for uniformly decelerating shear. ·····, quasi-steady; ——, 'standard' theory with two terms of the approximately quasi-steady expansions; – – –, 'standard' theory with only one term.

between the three choices for t_0 (equations ($10a$–c)). Again the second term in the quasi-steady series can be ignored for convenience, and in that case the three choices are equivalent to

$$t_0 = t_1 + k_2[9x/S(-t_1)]^{\frac{2}{3}} \qquad (17)$$

with

$$k_2 = 4C_0^2/\pi C_1^2 = 0{\cdot}175 \quad (\text{case } a),$$
$$k_2 = C_0 = 0{\cdot}187 \quad (\text{case } b),$$
$$k_2 = \tfrac{1}{4}\pi C_1^2 = 0{\cdot}200 \quad (\text{case } c).$$

The differences are small, and the numerical results confirm that this choice is not a crucial one; we subsequently adhere to the first, ($10a$).

Before presenting numerical results, we may note that the heat transfer can be calculated analytically for the case of uniformly decelerating shear when the second term in the quasi-steady series is ignored. This calculation provides a useful check on the numerical results, but has no intrinsic interest and will not be reproduced here.

The results are shown in figures 1 and 2. Figures 1(a) and (b) show graphs of $-\theta_y|_{y=0}$ against time for one particular position over the film, $x = 0{\cdot}5$. In each case the dotted curve is the quasi-steady solution. In figure 1(a) the three other curves represent the diffusive solution with t_0 given by (17) with $k_2 = 0{\cdot}175$ and with three different values of t_1 given by the three values of k_1 in (16). Note that only one term of the series (4) or (5) is included in each of the quasi-steady regions. The solid curve is for $k_1 = 1$, which can be regarded as the standard case since this is the only value for which a reasonable physical justification can be given; the broken curves are for $k_1 = 0{\cdot}5$ and $k_1 = 1{\cdot}5$. The discontinuities at the times of transition between quasi-steady and diffusive solutions are disturbingly large, especially at t_2, when a new quasi-steady solution takes over from the diffusive solution. The choice of t_0 is made to ensure some continuity at $t = -t_1$, but there is no freedom to take account of the increasing convection from the trailing edge until $t = t_2$; we shall have to accept that the present method leads to an underestimate of heat transfer for a period just after shear reversal. It can be seen that the lower value of k_1 leads to the smallest discontinuity, but that is solely because there is a shorter period of diffusion. We know from the previous discussion that the quasi-steady solution will be inaccurate between the times marked A and B and between C and D.

In figure 1(*b*) the dotted curve is again the quasi-steady solution. Here the three diffusive solutions represent three values of the constant k_2 in (17) ($k_1 = 1$ in each case): the solid curve is the 'standard' case with $k_2 = 0{\cdot}175$; the two broken curves are for $k_2 = 0{\cdot}15$ (upper) and $0{\cdot}20$ (lower). It is clear that these differences are not great, and little error will result from adhering to $k_2 = 0{\cdot}175$ (or (10*a*)). The smoothest-looking solution (at least at $t = -t_1$) is that which makes

$$\int_0^\infty \theta\, dy$$

continuous (equation (10*c*)), but this has no physical justification. To choose it would not necessarily be correct; it makes the heat-transfer curve smoother because the quantity made continuous is weighted more heavily towards the wall region than in the other cases.

The addition of the second term to the quasi-steady series makes a negligible difference to the diffusive part of the solution, but does of course affect the solution in the 'quasi-steady' regions. This is shown by the solid curves in figure 1(*b*). There is some reduction in the severity of the discontinuities at $t = -t_1$ and $t = t_2$, and therefore the second term is worth retaining, but the difference is slight.

The discontinuities are less apparent in the curves of total film heat transfer (Q, equation (12)) against time, as can be seen from figure 2. Once again the dotted curve represents the quasi-steady solution, the solid curve represents the standard solution with two terms of the quasi-steady series, and the broken curves show how taking only one term would alter the results. From the previous discussion we can expect an underestimate of heat transfer after the shear reversal, at $t = 0$; nevertheless we can clearly see that a considerable departure from quasi-steady behaviour is to be expected from just before reversal ($t = -0{\cdot}1$) to some considerable time after ($t = 1{\cdot}4$, say).

Sinusoidal oscillations

The form of shear variation most likely to be used in the unsteady calibration of shear measuring probes is a sinusoidal oscillation. If, in dimensional terms, the wall shear is

$$\hat{S}(\hat{t}) = S_0(1+\alpha_1 \cos \Omega \hat{t}),$$

then the dimensionless shear is

$$S(t) = 1+\alpha_1 \cos \omega_1 t, \tag{18}$$

where ω_1 is the dimensionless angular frequency, equal to $\Omega(l^2/S_0^2\,\kappa)^{\frac{1}{3}}$ (see (1) *et seq.*).

If $\alpha_1 > 1$, so that the shear reverses twice each cycle, the methods of this paper are applicable to the first reversal (with a suitable shift of time origin) as long as (7) has a solution for t_1 for each x in the range $(0, 1)$. They are also applicable to the second reversal as long as a quasi-steady phase intervenes after the first diffusive phase. If no reversed quasi-steady phase arises, the first diffusive solution must be continued throughout the second reversal, until the forwards quasi-steady solution takes over again sometime before the shear maximum. If $\alpha_1 < 0$, so that there is no shear reversal, then the diffusive solution is chosen to take over at the time when $-\lambda_1(t)$ first takes the value 0·5; if there is no such time, the (approximately) quasi-steady solution is valid throughout. If ω_1 is too large for the quasi-steady solution to be applicable anywhere, this procedure misleadingly indicates a short range of times (near that of maximum shear) when it is applicable. This can be checked by working out the coefficient of the third term in the quasi-steady series (Pedley 1972*a*), and if that exceeds about 2 at $t = 0$ and $x = 1$, i.e. if $\omega_1^2 \gtrsim 2\,(1+\alpha_1)^{\frac{7}{3}}/3^{\frac{5}{3}}\alpha_1$, the quasi-steady solution is not accurate, and the present methods should not be used.

The procedure for computing the film heat transfer as a function of t is straightforward, and follows almost exactly the same lines as the wall shear calculations of Pedley (1976). No further details will be given, therefore, and we proceed straight to the results. These are presented in figures 3(*a*), (*b*) and (*c*) for three

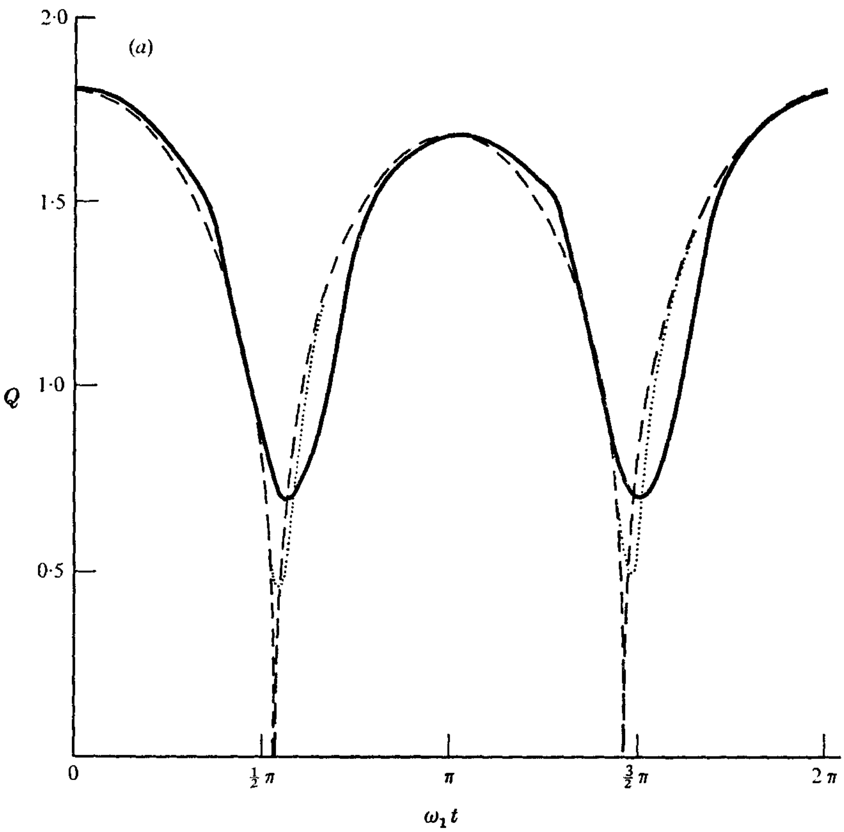
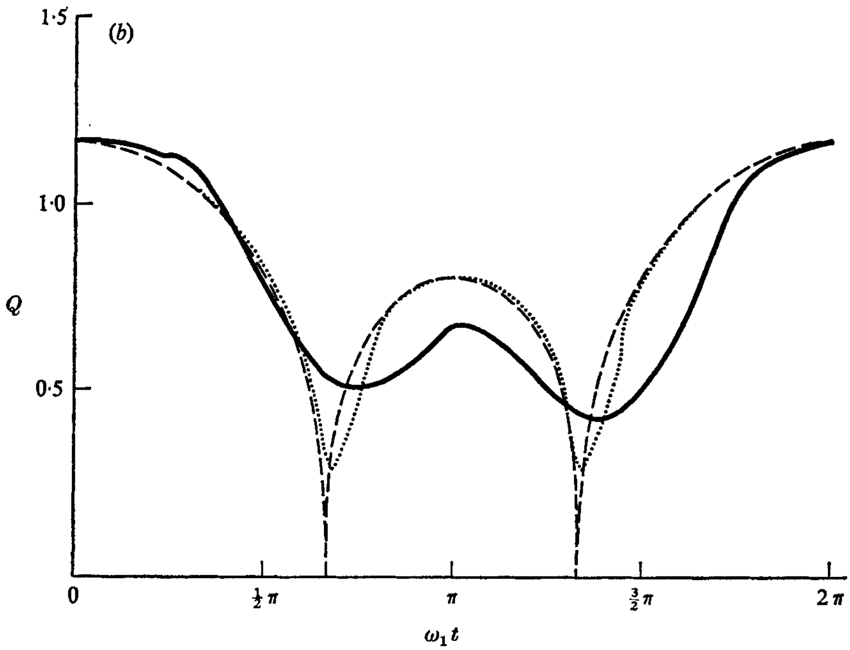


FIGURE 3(*a*). For legend see next page.



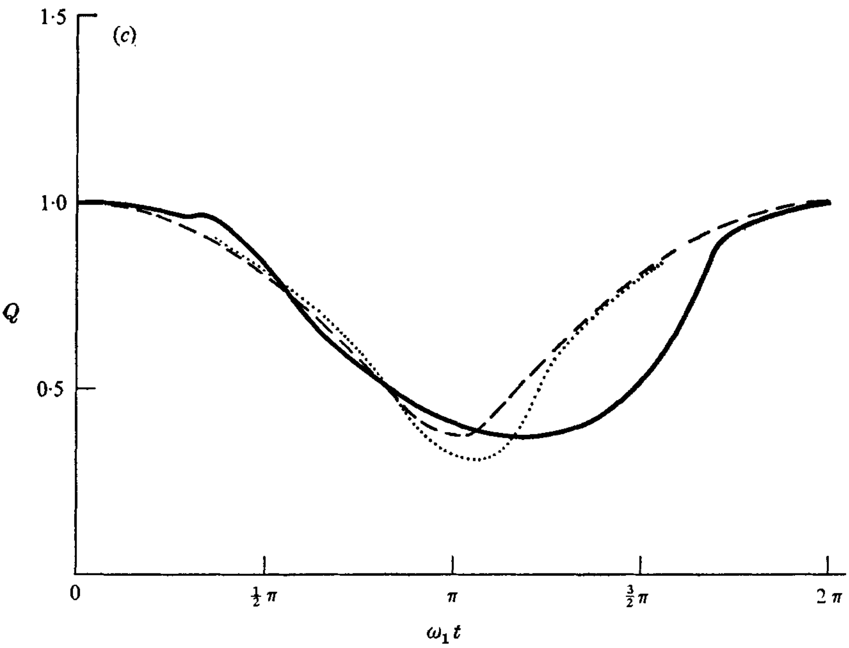


FIGURE 3. Graphs of Q against $\omega_1 t$ for oscillatory shear (equation (18)). – – –, quasi-steady; ····, $\omega_1 = 0{\cdot}1$; ——, $\omega_1 = 1{\cdot}0$. (a) $\alpha_1 = 10{\cdot}0$, (b) $\alpha_1 = 2{\cdot}0$, (c) $\alpha_1 = 0{\cdot}9$.

values of α_1 ($= 10{\cdot}0$, $2{\cdot}0$ and $0{\cdot}9$) and for two values of ω_1 ($= 1{\cdot}0$ and $0{\cdot}1$); the quasi-steady curve is shown in each case for comparison. Higher values of frequency were not chosen because the theory rapidly becomes inapplicable; however, $\omega_1 = 1{\cdot}0$ corresponds to a frequency of about 10 Hz for a 0·1 mm film in water or blood at a mean shear rate of $100\,s^{-1}$, which is considerably less than that usually encountered in the cardiovascular system so ω_1 will normally be $< 1{\cdot}0$.

The results for $\alpha_1 = 10{\cdot}0$ (figure 3*a*) show that in this case the two reversals are independent, with a period of approximately quasi-steady (but reversed) heat transfer in between. Each reversal looks like the single reversal of figure 2, with both t and Q appropriately rescaled. For $\alpha_1 = 2{\cdot}0$ (figure 3*b*), however, reversed quasi-steady heat transfer is not attained between the two reversals, at least when $\omega_1 = 1{\cdot}0$. This indicates that some part of the film (near $x = 0$) experiences purely diffusive heat transfer for the whole of the reversed phase. Figure 3(*c*) is an example of a non-reversing case ($\alpha_1 = 0{\cdot}9$), in which diffusion must nevertheless take over on much of the film for a part of each cycle (of course, the region very near $x = 0$ will always have quasi-steady heat transfer in this case). All the results of figure 3 show a slight phase lag behind the quasi-steady solution at periods of maximum shear (associated entirely with the second term in the solutions (4) and (5)), and a rather larger phase lag near times of minimum heat transfer. However, the latter may merely reflect the inaccuracy inherent in the method.

4. Application to the hot-film anemometer

In this section we endeavour to reproduce theoretically the conditions of the unsteady calibration experiments of Seed & Wood (1970) and of Clark (1974), each of whom used a probe like that depicted schematically in figure 4 (they used other probes as well, but details of the output as it varies throughout the cycle were given only for this). The film is embedded in an insulating substrate and mounted on the surface of a hypodermic needle which is bent into an L-shape so that the point can be aligned with the flow after insertion through an artery wall. The calibration experiments were performed in water; Seed & Wood oscillated the probe in a steady mean flow; Clark did the same at frequencies above 15 Hz, but held the probe fixed in an oscillating flow at lower frequencies. The relevant dimensions (as marked in figure 4) and operating conditions in each case are given in table 1. The flow velocity $\hat{U}(\hat{t})$ past the probe was in each case taken to be sinusoidal, with angular frequency Ω:

$$\hat{U}(\hat{t}) = U_0(1 + \alpha \cos \omega t), \tag{19}$$

where $t = \hat{t}L/U_0$, and $\omega = \Omega L/U_0$ is the dimensionless frequency; note that $\omega X_0/L$ is the same as the Strouhal number St defined by Clark (1974).

In order to apply the theory of this paper and of Pedley (1976) we must make a number of assumptions, as follows:

(i) The flow field on the probe is effectively two-dimensional.

(ii) The flow over the probe resembles that over a flat plate (i.e. has zero pressure gradient in steady flow).

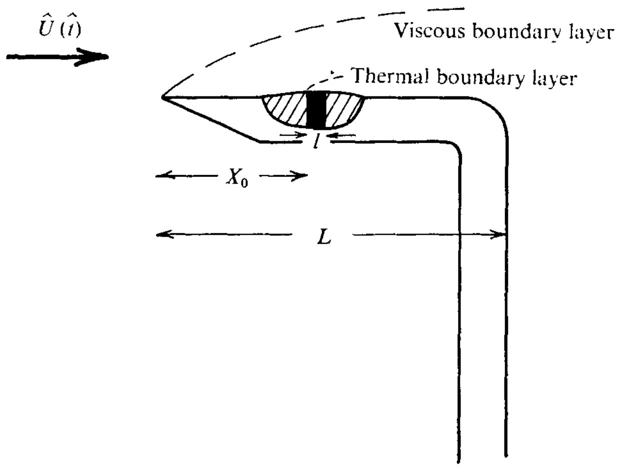


FIGURE 4. Sketch of hot-film anemometer probe. The dark rectangle represents the film; the shaded region around it represents the insulating substrate. Lengths l, L, X_0 are defined.

	l	X_0	L	Temperature of ambient water	Prandtl number ν/κ	β (see text)
Seed & Wood (1970)	0·01 cm	0·15 cm	0·3 cm	37 °C	4·6	14·0
Clark (1974)	0·02 cm	0·25 cm	0·5 cm†	20 °C	6·9	9·6

† Clark did not report the value of L, but since he did not use the probe in reversing flow it is irrelevant. We choose $L = 0{\cdot}5$ cm so that X_0/L can be taken equal to 0·5 in each case.

TABLE 1

(iii) The distance X_0 of the film from the probe's leading edge (and that from the trailing edge, $L - X_0$) is large enough for viscous boundary-layer theory to be applicable in calculating the shear on the film.

(iv) The film length l is sufficiently large for thermal boundary-layer theory to be applicable in calculating heat transfer.

(v) The temperature field over the film is effectively two-dimensional.

(vi) The 'thermal wake' is negligible; i.e. the fact that already heated fluid is carried back over the film during shear reversal can be ignored.

(vii) The thermal boundary layer over the film is sufficiently thin compared with the viscous boundary layer at that location that curvature of the velocity profile does not influence heat transfer.

(viii) The film length l is sufficiently small for the shear over it to be independent of x.

(ix) The normal velocity over the film is too small to affect heat transfer.

(x) Free convection is negligible.

(xi) Conduction in the substrate is negligible.

Assumptions (viii)–(xi) were adequately justified by Pedley (1972*a*) [(viii) and

(ix) are in fact equivalent, requiring $l/X_0 \ll 1$]; the validity of the rest is discussed in the next section.

The dimensionless shear $S(t)$ on the film can now be derived directly from equations (14*a*–*c*) of Pedley (1976), where it is referred to as τ, in the manner described in § 3 of that paper. The quantity x there is equal to X_0/L, and is taken to be 0·5 in each case (see table 1). Then the methods of this paper can be used to predict the heat transfer. In order that no rescaling of time is required between the output of one paper and the input of the next, we redefine the time scale $\hat{t}_0$ (equation (1)) to be equal to L/U_0. The scale S_0 for shear is then $(U_0^3/\nu L)^{\frac{1}{2}}$, so the quantity β in (2) and (3) takes the value (L/l) $(\kappa/\nu)^{\frac{1}{3}}$, which is also given in table 1 for each set of experiments. $S(t)$ must be multiplied by β wherever it occurs in the theory of § 2.

The results will be presented in terms of the velocity which would be inferred from the heat-transfer measurements if the quasi-steady relationship between velocity and heat transfer were assumed (Clark (1974, figure 9) presented his measurements in this way). The complete cycle is examined for five cases, as listed in table 2. Seed & Wood (1970) reported some measurements in reversing flow and some in non-reversing flow; however their data were presented in terms of the *ratio* between actual probe output and the output which would have been measured at the known instantaneous velocity in steady flow. When the latter becomes very small, inferring velocities from their data becomes very inaccurate. Therefore only two of their cases are chosen. Clark did not examine reversing flow, but in two of the three cases he presented the shear on the probe did reverse, and the experiments provide a reasonable test of the theory. The two Seed & Wood cases are presented in figures 5(*a*) and (*b*); the three Clark cases are presented in figures 6(*a*), (*b*) and (*c*). In each case the actual velocity wave form is also shown (so, in figure 5(*b*), is the rectified form of it which a perfectly quasi-steady anemometer would measure).

Figures 5(*a*) and (*b*) show reasonable qualitative agreement between the theory and Seed & Wood's experiments, especially near the points of flow reversal, although in each case the apparent velocity inferred from their data when the actual velocity is very low is enormous, and must be regarded as uncertain. Not enough experimental points were given in each cycle to constitute a good test of the theory. In the approximately quasi-steady regimes the predictions show a slight phase lead over the experiments, which rather follow the exactly quasi-steady curve. This phase lead comes from the phase lead of wall shear over free stream velocity. Apparently the heat transfer in practice lags behind the wall shear more than is predicted by this theory (as also remarked by Pedley 1972*a*).

Figure 6(*a*) shows an example in which the shear stress does not approach close enough to zero for a diffusive regime to appear at all. It is included in order to show that the phase lead of theory over experiment is quite pronounced here too (about $\frac{1}{8}\pi$). Figure 6(*b*) shows a case in which the flow does not reverse but the shear does. A comparison between the theoretical curve and the open triangles shows excellent agreement, apart from a slight underestimate of the maximum heat transfer in the approximately quasi-steady regime. Unfortunately,

Author(s)	α	ω	Flow reversal?	Shear reversal?	Figure
Seed & Wood	0·98	0·28	No	Yes	5(*a*)
Seed & Wood	2·8	0·75	Yes	Yes	5(*b*)
Clark	0·31	0·44	No	No	6(*a*)
Clark	0·56	0·80	No	Yes	6(*b*)
Clark	0·68	0·80	No	Yes	6(*c*)

TABLE 2

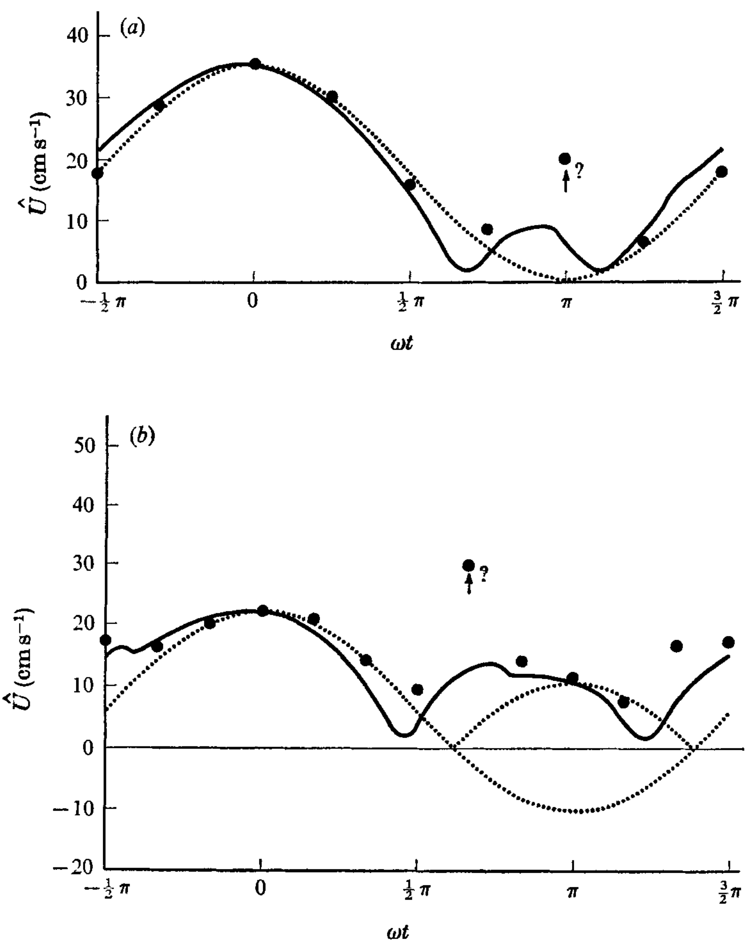


FIGURE 5. Dimensional velocity $\hat{U}$ (equation (19)) plotted against ωt for two cases from Seed & Wood's (1970) experiments. · · · · ·, actual velocity, which would be measured by a perfectly quasi-steady instrument (including the rectified signal during flow reversal in (*b*)); ——, predictions of the velocity which would be recorded by the instrument; ●, measurements. (*a*) $\alpha = 0{\cdot}98$, $\omega = 0{\cdot}28$; (*b*) $\alpha = 2{\cdot}8$, $\omega = 0{\cdot}75$.

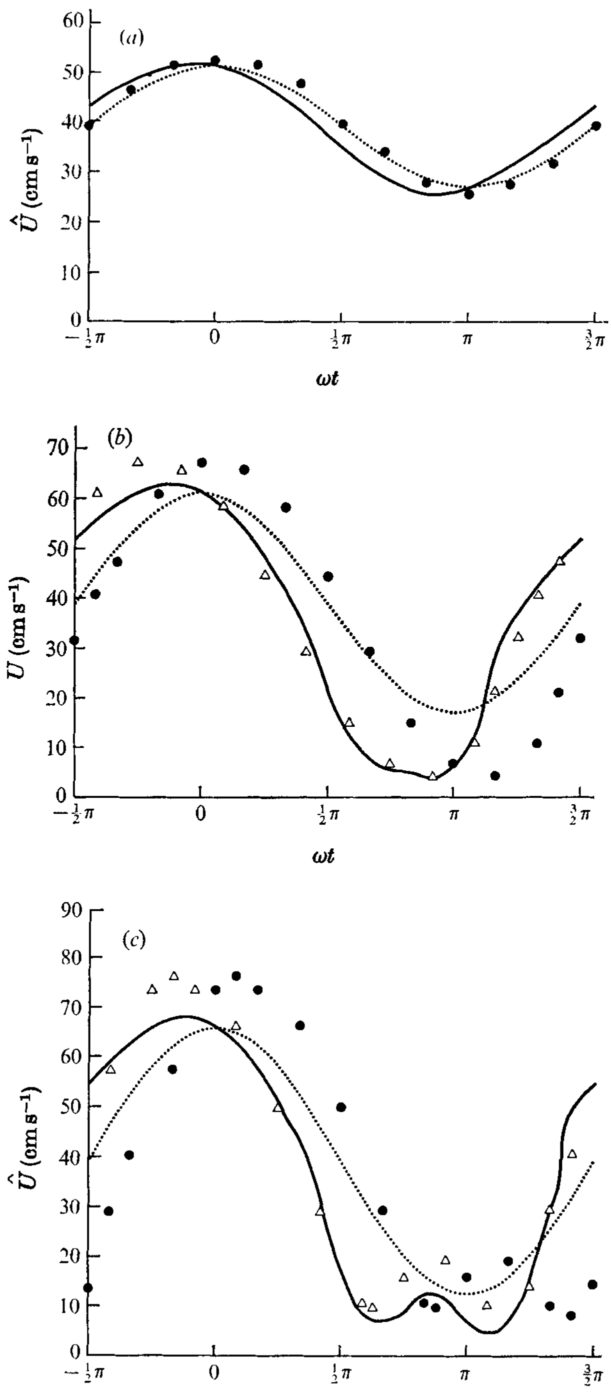


FIGURE 6. As figure 5 for three cases from Clark's (1974) experiments. (*a*) $\alpha = 0{\cdot}31, \omega = 0{\cdot}44$; (*b*) $\alpha = 0{\cdot}56$, $\omega = 0{\cdot}80$; (*c*) $\alpha = 0{\cdot}68$, $\omega = 0{\cdot}80$. The open triangles in (*b*) and (*c*) are the measured points given a phase lead of $\frac{1}{4}\pi$.

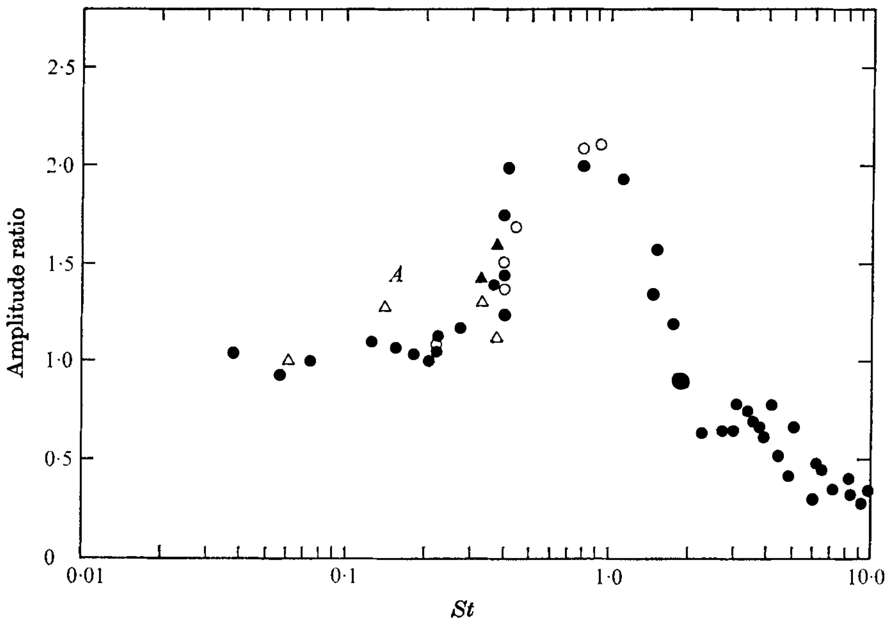


FIGURE 7. Ratio of amplitude of probe output to amplitude of actual oscillating velocity plotted against Strouhal number St ($= 0{\cdot}5\omega$). ●, Clark's experiments; ▲, Seed & Wood's experiments; ○, △, present theory.

however, the open triangles are not the experimental points, which are in fact represented by closed circles; the open triangles are the same points given a phase shift of $\frac{1}{4}\pi$. In other words the phase lead, remarked on above, has become considerable, but the shape of the heat-transfer response, especially near minimum velocity, is very well predicted. Figure 6(c) shows another case of even larger amplitude non-reversing flow. Again the agreement between theory and the open triangles is quite good (apart from underestimating the heat-transfer maxima), and again these represent a phase lead of $\frac{1}{4}\pi$ over the experimental points. Note that the $\frac{1}{8}\pi$ in figure 6(a) and the $\frac{1}{4}\pi$ in figures 6(b) and (c) represent an approximately constant *time* lead, independent of frequency. Possible reasons for the phase lead are discussed in the next section.

In most of the cases he studied, Clark did not calculate the apparent velocity throughout the cycle, merely at the times of maximum and minimum probe output. He then plotted the ratio of the apparent velocity amplitude to the actual velocity amplitude against the Strouhal number St ($= 0{\cdot}5\omega$); a value significantly different from 1 indicated that the quasi-steady calibration was inapplicable. Figure 7 shows his results (closed circles) together with the predictions of the present theory for six of his cases (open circles) and four of Seed & Wood's (open triangles); the two closed triangles are Seed & Wood's experimental results, corresponding to the open triangles at the same values of St. In cases where the heat transfer shows a second maximum (as in figure 5(b) or 6c), this is interpreted as measuring a negative velocity, even when the free stream does not reverse. The results show good agreement between experiment and theory up to a value of St of about 1·0; above that value the theory is not directly applicable. There is considerable scatter in both sets of results at any given value of St, especially around 0·3, which is associated with the fact that the result depends on amplitude as well as on frequency. Nevertheless, the theory confirms the experimental finding that the quasi-steady calibration cannot be used for $St \gtrsim 0 \cdot 2$. Even for smaller values of St, the theoretical results (e.g. point A on figure 7) underline the fact that the quasi-steady calibration will break down if the amplitude of the oscillation is sufficiently large that the shear on the probe reverses.

5. Discussion of assumptions

Here we investigate assumptions (i)–(vii) (pp. 525–526), with the aim of elucidating the discrepancy between theory and experiment, particularly the phase lead in figures 5 and 6. It was predicted in § 3 that the phase lag of heat transfer behind wall shear is not large for values of ω_1 up to 1·0 (equivalent to the range of ω considered in § 4), while the phase lead of wall shear over free-stream velocity approaches $\frac{1}{4}\pi$ for these values (Pedley 1972*b*). Hence our discussion should centre on whether in practice (*a*) the shear phase lead would be reduced or (*b*) the heat transfer lag would be increased.

(i) The probes are approximately cylindrical in cross-section, and at the station occupied by the film the steady boundary-layer thickness is about 0·01–0·02 cm for velocities of 20–80 cm s^{-1}, while the cylinder diameter is about 0·045 cm (in Clark's experiments). This means that the quasi-steady wall shear rate is greater by about 30 % than on a flat plate (Rosenhead 1963, p. 450). The argument by which the unsteady shear is predicted to have a phase lead over the outer velocity, because the flow near the wall responds more readily to the unsteady pressure gradient than that far away, is unaffected by the cylindrical geometry. If the correction to the quasi-steady shear is unchanged, the difference between the one- and two-term expansions could be significantly reduced. The phase lead could be reduced from $\frac{1}{4}\pi$ to about $\frac{1}{6}\pi$ in the case of figures 6 (*b*) and (*c*); this does not explain the whole discrepancy, but clearly deserves further investigation.

(ii) If the probe is yawed, or not quite cylindrical, then the flow may be subjected to a pressure gradient. Pedley (1972*b*) showed that a favourable pressure gradient cuts down the phase lead of shear over outer velocity. For two-dimensional flow impinging symmetrically on a 90° wedge, the relative magnitude of the term producing the phase lead (the second term of the expansion whose first term is quasi-steady) is reduced by about 75 %. However, it is inconceivable that the present probes induce such a strong pressure gradient (even a 10° wedge would have a stronger effect), and this factor cannot explain the discrepancy.

(iii) The Reynolds number based on the mean velocity and the distance of the film from the leading edge of the probe, $\overline{Re} = U_0 X_0/\nu$, is only about 1000 in Clark's experiments (figure 6). Thus viscous boundary-layer theory may not be adequate to predict the shear over the film, but it is not easy to see how this inadequacy would reduce the phase lead of the shear over the outer velocity. Indeed, since the viscous region would be thicker than in boundary-layer theory one might expect the phase difference to be greater if anything. Another argument against this as the explanation of the discrepancy is the fact that one would expect the effect to increase as $\overline{Re}$ falls, whereas the discrepancy is less in Seed & Wood's experiments (figure 5), where $\overline{Re} \approx 450$.

(iv) Thermal boundary-layer theory is inadequate for predicting steady heat transfer from a hot film if the Péclet number $Pe = S_0 l^2/\kappa$ is less than about 400 (Springer 1974, Table 1). For Clark's experiments the mean Péclet number is about 1000, but for Seed & Wood's it is only about 200, so the effect of departures from boundary-layer theory should be considered. The problem is not susceptible to immediate intuitive solution. On the one hand the presence of axial diffusion increases the effective length of the thermal boundary layer, which would indicate a greater time lag between heat transfer and wall shear. On the other hand in steady flow it increases the net heat-transfer rate in a manner only slightly dependent on the flow, which would suggest that the effect of unsteadiness would be less. Once again, however, if this effect were responsible for the discrepancy, one would expect a greater discrepancy in Seed & Wood's experiment than in Clark's, not a smaller.

(v) The hot films in both sets of experiments are only about 2·5 times longer in the cross-stream direction than in the streamwise direction, so lateral end effects are likely to be important. Again, the heat transfer in steady flow is increased by the lateral diffusion in a manner which is almost flow-independent, but the effect on the phase is difficult to assess. It probably does increase the phase lag, in the same way as diffusion through the substrate increases the phase lag though an increase in 'thermal inertia' (Bellhouse & Schultz 1967). This is the second of the possible explanations which cannot be ruled out as the explanation of the phase lag, and clearly warrants further theoretical and experimental study.

(vi) The 'thermal wake' effect would, if important, cause the actual heat transfer to be reduced below the predicted heat transfer during shear reversal, not increased as indicated in figure 6(*c*). It can have no effect on the phase near the maximum velocity.

(vii) In steady flow over a flat plate, the curvature of the velocity profile at the wall is zero (in fact $u \propto \alpha_2 \zeta - \frac{1}{24}\alpha_2^2 \zeta^5 + \dots$, where ζ is the Blasius similarity variable and $\alpha_2 \approx 0{\cdot}47$) and therefore has no effect on heat transfer. In unsteady flow, however, the curvature is non-zero because the pressure gradient is non-zero, and we have

$$u = U(t)\,(0{\cdot}47\zeta - \dots) + x\dot{U}U^{-1}(1{\cdot}2\zeta - \zeta^2 + \dots)$$

(see, for example, equation (5) of Pedley 1976). The pressure-gradient term has the opposite sign to the perturbation in wall shear, and thus tends to counteract its effect, reducing the phase lead of prediction over experiment. However, the calculated difference turns out to be very small in the present examples, and cannot account for the discrepancy.

Of all the potential fluid-mechanical reasons for the unwanted phase lead, the only possibilities not yet ruled out are three-dimensional effects in the velocity and the temperature fields, and we cannot be certain of their importance without further research. These apart, the explanation can only be in probe construction or electronics, and these are unlikely because of the great care which both Clark and Seed & Wood took to eliminate such artifacts. Thus the phase discrepancy remains a mystery, but should not be allowed to obscure the fact that the theory agrees very well with experiment as far as the amplitude response (figure 7) and the general shape of the response throughout the cycle (see especially figure 6*b*) are concerned.

Finally we should remember that both probes were designed for use in blood, and the calibration experiments described here were performed in water (both Clark and Seed & Wood also used blood, but did not give the complete cycle response in that case). Blood has a similar thermal diffusivity to water at the same temperature, but a significantly greater viscosity, by a factor of about 4 if the viscosity of whole blood is chosen, or by a factor of about 1·8 if it is plasma viscosity which is relevant as suggested by Clark (1974). Thus the absolute value of the shear on the film will be decreased, but its phase lead over the outer flow is unaffected; the relationship of the heat transfer to the shear will also be unaffected as long as the theory of this paper is still applicable. The effective Reynolds number is reduced in blood, so of the above assumptions it is only (i) and (iii) which are called into question. Calibration studies for any probe must cover the same Reynolds number range as will be experienced in blood, in case the breakdown of these assumptions becomes more important. Furthermore blood is a particulate suspension, and its microstructure may well have an important effect on heat transfer when the thickness of the thermal boundary layer becomes as small as the red-cell diameter or spacing (about 10 μm), which is the case for these experiments (see Seed & Wood 1970, figure 8). Thus calibration ought to be done in blood as well as in water.

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